

HOW LIKELY ARE TWO INDEPENDENT RECURRENT EVENTS TO OCCUR SIMULTANEOUSLY DURING A GIVEN TIME?

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ABSTRACT. We determine the probability P of two independent events A and B , which occur randomly n_A and n_B times during a total time T and last for t_A and t_B , to occur simultaneously at some point during T . Therefore we first prove the precise equation

$$P^* = \frac{t_A + t_B}{T} - \frac{t_A^2 + t_B^2}{2T^2}$$

for the case $n_A = n_B = 1$ and continue to establish a simple approximation equation

$$P \approx 1 - \left(1 - n_A \frac{t_A + t_B}{T}\right)^{n_B}$$

for any given value of n_A and n_B . Finally we prove the more complex universal equation

$$P = 1 - \frac{(T^+ - t_A n_A - t_B n_B)^{n_A + n_B}}{(T^+ - t_A n_A)^{n_A} (T^+ - t_B n_B)^{n_B}} \pm E^\pm,$$

which yields the probability for A and B to overlap at some point for any given parameter, with $T^+ := T + \frac{t_A + t_B}{2}$ and a small error term E^\pm .

1. Introduction

Let us consider two independent and recurring events A and B , which take place during a total time T . The events occur exactly n_A and n_B times randomly during this total time and last for t_A and t_B until the next event occurrence may happen. Since both events occur independently, they may eventually take place simultaneously at some point during T – i.e. they overlap. This may happen in case an occurrence

- (1) of A and B take place at the same moment,
- (2) of A takes places while an occurrence of B still takes place or
- (3) of B takes place while an occurrence of A still takes place.

We divide all time parameters in discrete units $\Delta \rightarrow 0$ and let P denote the probability for at least one overlap – i.e. it exists at least one time unit, during which both events take place. We will derive expressions to determine P under given parameters T , t_A , t_B , n_A and n_B . The following examples and notes clarify the setting and will be revisited later:

Example I — John works inside his office for 2 hours. A blue car will occur 10 times on the nearby street and remains visible for 1 minute each time. John looks outside 5 times for 3 minutes each. How likely is John going to see a blue car?: $T = 120 \text{ min}$, $t_A = 3 \text{ min}$, $t_B = 1 \text{ min}$, $n_A = 5$, $n_B = 10$ and we are about to determine $P(120, 3, 1, 5, 10)$.

Example II — We choose a total time of 3 seconds. Both events happen only once for 1 second during this time: How likely are both events to overlap at some point?: $T = 3 \text{ sec}$, $t_A = 1 \text{ sec}$, $t_B = 1 \text{ sec}$, $n_A = 1$, $n_B = 1$ and we are about to determine $P(3, 1, 1, 1, 1)$.¹

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¹Note: Since all time values are given with same unit, we will stop to note the unit in further calculations.

Definitions and Notes:

- (a) *If the duration time of A and B differs, let always A denote the event with the longer duration time: $t_A \geq t_B$.*
- (b) *The duration of the event-overlap is not relevant – i.e. an overlap of a few minutes or a millisecond are both counted as valid overlap.*
- (c) *We consider a possible overlap with the end of the total time as more natural and thus valid - e.g. John may start to look outside 4 seconds before the 120 minutes total time are over.*
- (d) *Both events occur precisely n_A and n_B times - we do not perceive these parameters as random in the proofs. When the events occur is the only random aspect first. We remark in Section 6 how the equation changes with n_A and n_B as random.*
- (e) *Occurrences of the same event must not overlap with themselves – e.g. John must not start to look outside while he is already looking. Thus $T \leq t_A \cdot n_A$ implies $P = 1$.*

2. Notation

Overview of main notation:

A, B	– labels of the events
T	– total time
t_A, t_B	– duration of event A, B
n_A, n_B	– number of occurrences of event A, B during total time
P, \bar{P}	– probability for an (no) overlap
Δ	– infinitesimal unit, divides time in discrete pieces

Overview of specific notations for local purpose:

S_1, S_2	– $\sum_{i=0}^{\frac{t_B}{\Delta}-2} (\frac{t_A}{\Delta} + i)$, $\sum_{j=1}^{\frac{t_A}{\Delta}-1} (\frac{t_A}{\Delta} + \frac{t_B}{\Delta} - 1 - j)$
T'	– T/Δ
N_A, N_B	– $(\frac{t_A}{\Delta} - 1) \cdot n_A$, $(\frac{t_B}{\Delta} - 1) \cdot n_B$
M_A, M_B, M	– $T' - N_A - n_A$, $T' - N_B - n_B$, $T' - N_A - n_A - N_B - n_B$
T^+	– $T + \frac{t_A + t_B}{2}$
α_A, α_B	– $T + t_A - t_A n_A - t_B n_B$, $\alpha_A - \frac{t_A - t_B}{2}$
τ	– $\max\{t_A n_A, t_B n_B\}$

3. Precise Equation for $n_A = n_B = 1$

We start with deriving an equation for the case $n_A = n_B = 1$, since

- (1) this case has shown to occur unexpectedly often in applications
- (2) prepares main techniques for the proof of Theorem 3 and
- (3) the equation yields precise results without error term².

Let $P^*(T, t_A, t_B)$ denote the probability function for this case.

Theorem 1. *The probability³ of A and B to overlap at some point during the total time T in case of $n_A = n_B = 1$ is given by*

$$P^*(T, t_A, t_B) = \frac{t_A + t_B}{T} - \frac{t_A^2 + t_B^2}{2T^2}.$$

²although $|t_A - t_B| > 0$ (see Section 3)

³Note: Due to reading purpose we will use just 'A' instead of 'the A occurrence(s)' – same for B.

Lemma 1.1. *We remember the well known arithmetic summation formula*

$$1 + 2 + \dots + n = \sum_{k=1}^n k = \frac{1}{2}n \cdot (n + 1).$$

Proof. See [2]. □

Proof of Theorem. First, we divide the time in pieces of size Δ . Thus, we get T/Δ as limited number of positions at which the events may start, so there are $(T/\Delta)^2$ possible arrangements. By determining the number of positions $x_B(x_A)$ of B to overlap with A for every possible position x_A of A , we are able to obtain the probability P^* with

$$P^* = \lim_{\Delta \rightarrow 0} \frac{\sum_{x_A} x_B(x_A)}{(T/\Delta)^2}. \quad (3.1)$$

In general, if A is located somewhere in the center of T – i.e. in a sufficient distance to beginning and end of the total time – there are

$$\frac{t_A}{\Delta} + \frac{t_B}{\Delta} - 1 \quad (3.2)$$

possible positions x_B for B to overlap with A . This holds true for

$$\frac{T}{\Delta} - \frac{t_A}{\Delta} - \frac{t_B}{\Delta} + 1$$

positions x_A of A , since there are at $\frac{t_B}{\Delta} - 1$ positions (beginning of T) and at $\frac{t_A}{\Delta} - 1$ positions (end of T) less possible positions for B to overlap with A . Therefore (3.1) becomes

$$\begin{aligned} P^* &= \lim_{\Delta \rightarrow 0} \frac{\sum_{i=0}^{\frac{t_B}{\Delta}-2} \left(\frac{t_A}{\Delta} + i \right) + \left(\frac{t_A}{\Delta} + \frac{t_B}{\Delta} - 1 \right) \left(\frac{T}{\Delta} - \frac{t_A}{\Delta} - \frac{t_B}{\Delta} + 1 \right) + \sum_{j=1}^{\frac{t_A}{\Delta}-1} \left(\frac{t_A}{\Delta} + \frac{t_B}{\Delta} - 1 - j \right)}{(T/\Delta)^2} \\ &= \lim_{\Delta \rightarrow 0} \frac{\Delta^2 S_1 + (t_A + t_B - \Delta)(T - t_A - t_B + \Delta) + \Delta^2 S_2}{T^2}. \end{aligned} \quad (3.3)$$

By applying Lemma 1.1 the first sum simplifies to

$$\begin{aligned} \Delta^2 S_1 &= \Delta^2 \sum_{i=0}^{\frac{t_B}{\Delta}-2} \left(\frac{t_A}{\Delta} + i \right) = (t_B - \Delta)t_A + \Delta^2 \sum_{i=1}^{\frac{t_B}{\Delta}-2} i \\ &= (t_B - \Delta)t_A + \frac{1}{2}(t_B - \Delta)(t_B - 2\Delta). \end{aligned}$$

and the second sum to

$$\begin{aligned} \Delta^2 S_2 &= \Delta^2 \sum_{j=1}^{\frac{t_A}{\Delta}-1} \left(\frac{t_A}{\Delta} + \frac{t_B}{\Delta} - 1 - j \right) = (t_A - \Delta)(t_A + t_B - \Delta) - \Delta^2 \sum_{j=1}^{\frac{t_A}{\Delta}-1} j \\ &= (t_A - \Delta)(t_A + t_B - \Delta) - \frac{1}{2}t_A(t_A - \Delta). \end{aligned}$$

Inserting these results in (3.3) and considering $\Delta \rightarrow 0$ yields

$$\begin{aligned} P^* &= \frac{t_A t_B + \frac{1}{2}t_B^2 + (t_A + t_B)(T - t_A - t_B) + t_A(t_A + t_B) - \frac{1}{2}t_A^2}{T^2} \\ &= \frac{t_A t_B + \frac{1}{2}t_B^2 + (t_A + t_B)(T - t_B) - \frac{1}{2}t_A^2}{T^2} \\ &= \frac{t_A T + t_B T - \frac{1}{2}t_A^2 - \frac{1}{2}t_B^2}{T^2} = \frac{t_A + t_B}{T} - \frac{t_A^2 + t_B^2}{2T^2}, \end{aligned}$$

which concludes the proof. □

— Examples —

Example I (modified) — John looks outside once for 5 minutes and a blue car occurs once for 2 minutes during 1 hour. He will see a blue car with a probability of about 11.26 %:

$$P^*(60, 5, 2) = \frac{5+2}{60} - \frac{5^2+2^2}{2 \cdot 60^2} \approx 0.1126 .$$

Example II — Two events with $t_A = t_B = 1$ sec will overlap during a total time $T = 3$ sec with a probability of about 55.56 %:

$$P^*(3, 1, 1) = \frac{1+1}{3} - \frac{1^2+1^2}{2 \cdot 3^2} \approx 0.5556 .$$

4. Approximation of Universal Equation

Before we derive a more complex universal equation (see Section 5), we will find an approximation, which shall apply for any given n_A and n_B .

Theorem 2. *The probability of A and B to overlap at some point during the total time T, with A and B occurring n_A and n_B times for t_A and t_B , can be approximated by*

$$P(T, t_A, t_B, n_A, n_B) \approx 1 - \left(1 - n_A \frac{t_A + t_B}{T}\right)^{n_B} .$$

Proof of Theorem. Since $P = 1 - \bar{P}$, we will establish an equation for the probability \bar{P} for the events not to occur at the same time at any point. Let $k \in \{0, 1, \dots, t_B - 1\}$ number the n_B occurrences of B . There are

$$\frac{T}{\Delta} - k \cdot \frac{t_B}{\Delta} \tag{4.1}$$

possible positions for the k -th B occurrence to start, but we consider $T \gg t_B$ and approximate (4.1) with T/Δ . Every B occurrence may overlap at about

$$\frac{t_A}{\Delta} + \frac{t_B}{\Delta}$$

positions (see (3.2)) with any occurrence of A . Thus we have for the probability \bar{p} of a B occurrence *not* to overlap at some point with any A occurrence

$$\bar{p} \approx 1 - \frac{n_A \cdot \left(\frac{t_A}{\Delta} + \frac{t_B}{\Delta}\right)}{T/\Delta} = 1 - n_A \frac{t_A + t_B}{T}$$

To find \bar{P} this has to be true for every occurrence of B and we get

$$P = 1 - \bar{P} = 1 - (\bar{p})^{n_B} \approx 1 - \left(1 - n_A \frac{t_A + t_B}{T}\right)^{n_B} ,$$

which confirms the approximation. □

— Example —

Example I — John will see a blue car with a probability of about 83.85 %:

$$P(120, 3, 1, 5, 10) \approx 1 - \left(1 - 5 \cdot \frac{3+1}{120}\right)^{10} \approx 0.8385 .$$

In Example I in Section 5 the error of this result will be shown to be insignificant.

5. Universal Equation

Theorem 3. *The probability of A and B to overlap at some point during the total time T, with A and B occurring n_A and n_B times for t_A and t_B , is given by*

$$P(T, t_A, t_B, n_A, n_B) = 1 - \frac{(T^+ - t_A n_A - t_B n_B)^{n_A + n_B}}{(T^+ - t_A n_A)^{n_A} (T^+ - t_B n_B)^{n_B}} \pm E^\pm$$

with $T^+ := T + \frac{t_A + t_B}{2}$ and a small error term

$$0 \leq E^\pm \leq \frac{(T^+ - t_A n_A - t_B n_B)^{n_A + n_B}}{(T^+ - t_A n_A)^{n_A} (T^+ - t_B n_B)^{n_B}} \cdot \left(\left(\frac{\alpha_A(\alpha_B + \tau)}{\alpha_B(\alpha_A + \tau)} \right)^{n_A + n_B} - 1 \right)$$

with $\alpha_A := T + t_A - t_A n_A - t_B n_B$, $\alpha_B := \alpha_A - \frac{t_A - t_B}{2}$ and $\tau := \max\{t_A n_A, t_B n_B\}$.

Lemma 3.1. *For constant $y \in \mathbb{N}$ we have*

$$\lim_{X \rightarrow \infty} \frac{(X + y)!}{X! X^y} = 1.$$

Proof. The above fraction can be rearranged to

$$\lim_{X \rightarrow \infty} \frac{(X + 1) \cdot \dots \cdot (X + y)}{X^y} = \lim_{X \rightarrow \infty} \prod_{l=1}^y \left(1 + \frac{l}{X} \right) = 1.$$

□

Lemma 3.2. *For constant $u, v \in \mathbb{R}$ with $u \leq v$ following inequality holds true:*

$$\frac{\alpha_B + u}{\alpha_A + u} \leq \frac{\alpha_B + v}{\alpha_A + v}.$$

Proof. Rearranging the inequality yields

$$\begin{aligned} (\alpha_A + v)(\alpha_B + u) &\leq (\alpha_A + u)(\alpha_B + v) \\ \alpha_A u + \alpha_B v &\leq \alpha_A v + \alpha_B u \\ \alpha_B(v - u) &\leq \alpha_A(v - u). \end{aligned} \tag{5.1}$$

Inequality (5.1) holds true since $v - u \geq 0$ and $\alpha_B := \alpha_A - \frac{1}{2}t_A + \frac{1}{2}t_B \leq \alpha_A$ due to Definition (a) in Section 1 that $t_A \geq t_B$. □

Proof of Theorem. Similar to the approximation we determine the probability \bar{P} for no overlap to occur. Therefore we divide the time in parts of size Δ again and count the number of possible arrangements of all A and B occurrences. There are

$$\frac{T}{\Delta} - \left(\frac{t_A}{\Delta} - 1 \right) \cdot n_A$$

positions for the n_A occurrences of A left, so that they do not overlap with themselves. Thus the number of possible arrangements of the A occurrences is given by

$$\binom{\frac{T}{\Delta} - \left(\frac{t_A}{\Delta} - 1 \right) \cdot n_A}{n_A} \tag{5.2}$$

and for the B occurrences by

$$\binom{\frac{T}{\Delta} - \left(\frac{t_B}{\Delta} - 1 \right) \cdot n_B}{n_B}. \tag{5.3}$$

The number of ways to order the n_A occurrences of A and the n_B occurrences of B is

$$\binom{n_A + n_B}{n_A}. \tag{5.4}$$

Similar to (5.2) and (5.3) we can express the number of ways to arrange the A and B occurrences per order of (5.4) as

$$\binom{\frac{T}{\Delta} - \left(\frac{t_A}{\Delta} - 1 \right) \cdot n_A - \left(\frac{t_B}{\Delta} - 1 \right) \cdot n_B}{n_A + n_B}. \tag{5.5}$$

Since the probability \bar{P} describes the ratio of arrangements without overlap⁴ to the total number of possible arrangements of the A and B occurrences, we have⁵

$$\bar{P} = \lim_{\Delta \rightarrow 0} \frac{\binom{n_A+n_B}{n_A} \left(\frac{T}{\Delta} - \left(\frac{t_A}{\Delta} - 1 \right) \cdot n_A - \left(\frac{t_B}{\Delta} - 1 \right) \cdot n_B \right)}{\left(\frac{T}{\Delta} - \left(\frac{t_A}{\Delta} - 1 \right) \cdot n_A \right) \left(\frac{T}{\Delta} - \left(\frac{t_B}{\Delta} - 1 \right) \cdot n_B \right)} = \lim_{\Delta \rightarrow 0} \frac{\binom{n_A+n_B}{n_A} \binom{T'-N_A-N_B}{n_A+n_B}}{\binom{T'-N_A}{n_A} \binom{T'-N_B}{n_B}}.$$

Rewriting the binomial coefficients and rearrange the fraction yields

$$\begin{aligned} \bar{P} &= \lim_{\Delta \rightarrow 0} \frac{(n_A+n_B)!(T'-N_A-N_B)!n_A!(T'-N_A-n_A)!n_B!(T'-N_B-n_B)!}{(T'-N_A)!(T'-N_B)!n_A!n_B!(n_A+n_B)!(T'-N_A-N_B-n_A-n_B)!} \\ &= \lim_{\Delta \rightarrow 0} \frac{(T'-N_A-N_B)!(T'-N_A-n_A)!(T'-N_B-n_B)!}{(T'-N_A)!(T'-N_B)!(T'-N_A-N_B-n_A-n_B)!} \\ &= \lim_{\Delta \rightarrow 0} \frac{(M+n_A+n_B)!M_A!M_B!}{(M_A+n_A)!(M_B+n_B)!M!}. \end{aligned} \quad (5.6)$$

Due to the limit we have $\Delta \rightarrow 0$ and thus $M_A, M_B, M \rightarrow \infty$. Therefore we are allowed to apply Lemma 3.1 on (5.6):

$$\begin{aligned} \bar{P} &= \lim_{\Delta \rightarrow 0} \frac{M!M^{n_A+n_B}M_A!M_B!}{M_A!M_A^{n_A}M_B!M_B^{n_B}M!} = \lim_{\Delta \rightarrow 0} \frac{M^{n_A+n_B}}{M_A^{n_A}M_B^{n_B}} \\ &= \lim_{\Delta \rightarrow 0} \frac{(T'-N_A-n_A-N_B-n_B)^{n_A+n_B}}{(T'-N_A-n_A)^{n_A}(T'-N_B-n_B)^{n_B}} \\ &= \lim_{\Delta \rightarrow 0} \frac{\left(\frac{T}{\Delta} - \frac{t_A}{\Delta}n_A - \frac{t_B}{\Delta}n_B \right)^{n_A+n_B}}{\left(\frac{T}{\Delta} - \frac{t_A}{\Delta}n_A \right)^{n_A} \left(\frac{T}{\Delta} - \frac{t_B}{\Delta}n_B \right)^{n_B}} \\ &= \frac{(T-t_An_A-t_Bn_B)^{n_A+n_B}}{(T-t_An_A)^{n_A}(T-t_Bn_B)^{n_B}}. \end{aligned}$$

This equation yields the probability in case that the occurrences may not overlap the total time. Since we defined in (c) in Section 1 the occurrences may overlap with the end of the total time, we have to extend the total time T by some $\delta > 0$:

$$\bar{P}_\delta := \frac{(T+\delta-t_An_A-t_Bn_B)^{n_A+n_B}}{(T+\delta-t_An_A)^{n_A}(T+\delta-t_Bn_B)^{n_B}}$$

Obviously, $t_B \leq \delta \leq t_A$ and we choose $\delta^* := \frac{t_An_A+t_Bn_B}{2}$. Therefore let us redefine

$$\bar{P} := \bar{P}_{\delta^*} \pm E^\pm = \frac{(T^+-t_An_A-t_Bn_B)^{n_A+n_B}}{(T^+-t_An_A)^{n_A}(T^+-t_Bn_B)^{n_B}} \pm E^\pm$$

with $0 \leq E^\pm$ as small error term. E^\pm approaches 0 if $|t_A - t_B|$ approaches 0 or if the ratio between T and the event time $t_An_A + t_Bn_B$ increases⁶

$$E^\pm \sim |t_A - t_B| \cdot \frac{t_An_A + t_Bn_B}{T}.$$

In order to determine the maximum error we consider, that the probability for an overlap decreases with longer T . Since we defined $t_A \geq t_B$ (see Definition (a) in Section 1), we have

⁴number of arrangements without overlap (5.5) per orders (5.4)

⁵see notation in Section 2

⁶Notation: ' \sim ' denotes 'proportional to'

$P_{\delta^*} - P_{t_A} \geq P_{t_B} - P_{\delta^*}$ and the maximum error is given by

$$\begin{aligned}
E^\pm &\leq P_{\delta^*} - P_{t_A} = (1 - \bar{P}_{\delta^*}) - (1 - \bar{P}_{t_A}) = \bar{P}_{\delta^*} \cdot \left(\frac{\bar{P}_{t_A}}{\bar{P}_{\delta^*}} - 1 \right) \\
&= \bar{P}_{\delta^*} \cdot \left(\left(\frac{\alpha_A}{\alpha_B} \right)^{n_A+n_B} \cdot \left(\frac{\alpha_B + t_B n_B}{\alpha_A + t_B n_B} \right)^{n_A} \cdot \left(\frac{\alpha_B + t_A n_A}{\alpha_A + t_A n_A} \right)^{n_B} - 1 \right) \\
&\leq \bar{P}_{\delta^*} \cdot \left(\left(\frac{\alpha_A}{\alpha_B} \right)^{n_A+n_B} \cdot \left(\frac{\alpha_B + \tau}{\alpha_A + \tau} \right)^{n_A+n_B} - 1 \right) \quad (\text{see Lemma 3.2}) \\
&= \bar{P}_{\delta^*} \cdot \left(\left(\frac{\alpha_A(\alpha_B + \tau)}{\alpha_B(\alpha_A + \tau)} \right)^{n_A+n_B} - 1 \right).
\end{aligned}$$

Finally the probability for an overlap is given by $P = 1 - \bar{P}$, which concludes the proof. \square

— Example —

Example I — John will see a blue car with a probability of about 85.46 ± 1.77 %. Since $T^+ = 120 + \frac{3+1}{2} = 122$, $\alpha_A = 120 + 3 - 3 \cdot 5 - 1 \cdot 10 = 98$ and $\alpha_B = 98 - \frac{3-1}{2} = 97$, we have

$$P(120, 3, 1, 5, 10) = 1 - \frac{(122 - 3 \cdot 5 - 1 \cdot 10)^{5+10}}{(122 - 3 \cdot 5)^5 (122 - 1 \cdot 10)^{10}} \pm E^\pm \approx 0.8546 \pm E^\pm$$

and, due to $\max\{3 \cdot 5, 1 \cdot 10\} = 3 \cdot 5$,

$$E^\pm \leq 85.46 \% \cdot \left(\left(\frac{98 \cdot (97 + 3 \cdot 5)}{97 \cdot (98 + 3 \cdot 5)} \right)^{5+10} - 1 \right) \approx 1.77 \%.$$

This confirms the precision of the approximation in Section 4.

Example II — Two events with $t_A = t_B = 1$ sec will overlap during a total time $T = 3$ sec at some point with a probability of about 55.56 %. Since $T^+ = 3 + \frac{1+1}{2} = 4$, we have

$$P(3, 1, 1, 1, 1) = 1 - \frac{(4 - 1 \cdot 1 - 1 \cdot 1)^{1+1}}{(4 - 1 \cdot 1)(4 - 1 \cdot 1)} \pm 0 \approx 0.5555 \pm 0$$

without any error due to $|t_A - t_B| = |1 - 1| = 0$. This result is consistent with Section 3.

6. Remark: n_A and n_B as random parameters

Instead of determining n_A and n_B as precise numbers of occurrences, it is more natural to define ρ_A and ρ_B , which represent the probability p for the event to occur in a given time s : $\rho := p/s$. In order to calculate the probability for a possible overlap, the number of expected occurrences $n = \rho \cdot T$ has to be calculated first. The universal equation would simply change to

$$P(T, t_A, t_B, \rho_A, \rho_B) = 1 - \frac{(T^+ - t_A T \rho_A - t_B T \rho_B)^{T \rho_A + T \rho_B}}{(T^+ - t_A T \rho_A)^{T \rho_A} (T^+ - t_B T \rho_B)^{T \rho_B}} \pm E^\pm.$$

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